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1999 J. Phys. A: Math. Gen. 32 6629

(http://iopscience.iop.org/0305-4470/32/38/302)

J. Phys. A: Math. Gen. **32** (1999) 6629–6642. Printed in the UK PII: S0305-4470(99)03798-1

Covariant Hamilton equations for field theory

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Received 22 April 1999, in final form 16 July 1999

Abstract. We study the relations between the equations of first-order Lagrangian field theory on fibre bundles and the covariant Hamilton equations on the finite-dimensional polysymplectic phase space of covariant Hamiltonian field theory. If a Lagrangian is hyperregular, these equations are equivalent. A degenerate Lagrangian requires a set of associated Hamiltonian forms in order to exhaust all solutions of the Euler–Lagrange equations. The case of quadratic degenerate Lagrangians is studied in detail.

1. Introduction

The finite-dimensional covariant Hamiltonian approach to field theory has been vigorously developed since the 1970s in its multisymplectic and polysymplectic variants [4, 5, 12]. In the framework of this approach, one deals with the following types of partial differential equations (PDEs): Euler–Lagrange and Cartan equations in Lagrangian formalism, Hamilton–De Donder equations in multisymplectic Hamiltonian formalism, covariant Hamilton equations and constrained Hamilton equations in polysymplectic Hamiltonian formalism. If a Lagrangian is hyperregular, all these PDEs are equivalent. This work addresses degenerate semiregular and almost-regular Lagrangians. From the mathematical viewpoint, these notions of degeneracy are particularly appropriate for the study of relations between the above-mentioned PDEs. From the physical one, Lagrangians of almost all field theories are of these types.

To formulate our results, let us characterize briefly the equations under consideration. Given a fibre bundle $Y \to X$ coordinated by (x^λ, y^i), let

$$L = \mathcal{L}\omega : J^1Y \to \overset{n}{\wedge} T^*X \qquad \omega = \mathrm{d}x^1 \wedge \dots \mathrm{d}x^n \qquad n = \dim X \tag{1}$$

be a first-order Lagrangian L on the jet bundle $J^1Y \to X$. The first variational formula provides the associated Euler–Lagrange equations. The Cartan equations characterize the variational problem on the repeated jet manifold J^1J^1Y for the Poincaré–Cartan form H_L. The Poincaré–Cartan form H_L yields the Legendre morphism $\hat{H}_L$ of J^1Y to the homogeneous Legendre bundle

$$Z_Y = T^*Y \wedge (\overset{n-1}{\wedge} T^*X) \tag{2}$$

which is the affine $\overset{n-1}{\wedge} T^*X$-valued dual of $J^1Y \to Y$ provided with the canonical exterior n-form Ξ_Y (18) [2,5]. If $\hat{H}_L(J^1Y)$ is an embedded subbundle of $Z_Y \to Y$, the pull-back of Ξ_Y yields the Hamilton–De Donder equations on $\hat{H}_L(J^1Y)$. If a Lagrangian L is almost regular, these equations are quasi-equivalent to the Cartan equations, i.e., there is a surjection of the set of solutions of the Cartan equations onto that of the Hamilton–De Donder equations [5].

A Lagrangian L yields the Legendre map $\hat{L}$ of J^1Y to the Legendre bundle

$$\Pi = \overset{n}{\wedge} T^*X \underset{Y}{\otimes} V^*Y \underset{Y}{\otimes} TX \tag{3}$$

equipped with the holonomic coordinates $(x^\lambda, y^i, p^\lambda_i)$. Π is provided with the canonical polysymplectic form Ω (22), and is seen as a momentum phase space of fields [3, 6, 7, 9, 11]. We have the one-dimensional affine bundle

$$\pi_{Z\Pi} : Z_Y \to \Pi. \tag{4}$$

Given any section h of $Z_Y \to \Pi$, the pull-back

$$H = h^*\Xi_Y = p^\lambda_i \, \mathrm{d}y^i \wedge \omega_\lambda - \mathcal{H}\omega \qquad \omega_\lambda = \partial_\lambda \rfloor \omega \tag{5}$$

is a Hamiltonian form on Π [2, 4, 12]. This is the Poincaré–Cartan form of the Lagrangian

$$L_H = (p^\lambda_i y^i_\lambda - \mathcal{H})\omega \tag{6}$$

on the jet manifold $J^1\Pi$. The associated Euler–Lagrange equations are the covariant Hamilton equations (31*a*), (31*b*). Every Hamiltonian form H (6) yields the Hamiltonian map $\hat{H}$ (29) of Π to J^1Y.

The results of this paper demonstrate that polysymplectic Hamiltonian formalism can provide an adequate description of degenerate field systems which do not necessarily possess gauge symmetries.

We show that, if $r : X \to \Pi$ is a solution of the covariant Hamilton equations for a Hamiltonian form H associated with a semiregular Lagrangian L and if r lives in the Lagrangian constraint space $\hat{L}(J^1Y)$, then $\hat{H} \circ r$ is a solution of the Cartan equations for L, while the projection s of r onto Y is that of the Euler–Lagrange equations. The converse assertion is more intricate. One needs a complete set of associated Hamiltonian forms in order to exhaust all solutions of the Euler–Lagrange equations (but not the Cartan equations). Given a solution s of the Euler–Lagrange equations, $\hat{L} \circ J^1s$ is a solution of the Hamilton equations for an associated Hamiltonian form H iff

$$\hat{H} \circ \hat{L} \circ J^1s = J^1s. \tag{7}$$

If a solution $\overline{s}$ of the Cartan equations provides the solution $\hat{L} \circ \overline{s}$ of covariant Hamilton equations, its projection s on Y is a solution of the Euler–Lagrange equations.

In view of these relations, one may conclude that the covariant Hamilton equations contain additional conditions in comparison with the Euler–Lagrange and Cartan equations. In the case of an almost-regular Lagrangian, we can introduce the constrained Hamilton equations which are weaker than the Hamilton equations restricted to the Lagrangian constraint space [3, 4, 13]. They are equivalent to the Hamilton–De Donder equations and, consequently, are quasi-equivalent to the Cartan equations.

We provide the detailed analysis of degenerate quadratic Lagrangian systems, appropriate for application to many physical models. Given a quadratic Lagrangian L, we find a complete set of associated Hamiltonian forms. The key point is the splitting of J^1Y into the dynamic sector and the gauge one coinciding with the kernel of the Legendre map $\hat{L}$. As a consequence, one can separate a part of the Hamilton equations independent of momenta which play the role of gauge-type conditions, while the rest equations restricted to the Lagrangian constraint space coincide with the constrained Hamilton equations, and are quasi-equivalent to the Cartan equations. We observe that the main features in gauge theory are not directly related to the gauge invariance condition, but are common in all field models with degenerate almost-regular quadratic Lagrangians. The important peculiarity of the Hamiltonian description of these models lies in the fact that, in comparison with a Lagrangian L, any associated Hamiltonian form H and the Lagrangian L_H (6) contain gauge fixing terms. Moreover, one can find a complete set of non-degenerate Hamiltonian forms associated with a degenerate quadratic Lagrangian, that is essential for quantization.

The plan of the paper is as follows. Section 2 presents some technical preliminaries. In section 3, the relations between Euler–Lagrange, Cartan and Hamilton–De Donder equations are summarized in a form suitable for our purposes. Section 4 provides a brief exposition of covariant Hamiltonian dynamics. In comparison with our previous works, we use the Lagrangian L_H (6) as a convenient tool in order to introduce the covariant Hamilton equations. Section 5 is devoted to the above-mentioned relations between degenerate Lagrangian and Hamiltonian systems. Degenerate quadratic Lagrangian systems are studied in section 6.

2. Technical preliminaries

All maps throughout the paper are smooth, while manifolds are real, finite-dimensional, Hausdorff, second-countable and connected. A base manifold X is oriented.

Let us recall some notions [4, 10, 14]. Given a fibre bundle $Y \to X$, the s-order jet manifold J^sY is endowed with the adapted coordinates (x^λ, y^i_Λ), $0 \leqslant |\Lambda| \leqslant s$, where Λ is a symmetric multi-index $(\lambda_k \ldots \lambda_1)$, $|\Lambda| = k$. The repeated jet manifold J^1J^1Y is coordinated by $(x^\lambda, y^i, y^i_\lambda, \hat y^i_\lambda, y^i_{\lambda\mu})$.

Exterior forms ϕ on J^sY, $s = 0, 1, \ldots$, are naturally identified with their pull-backs onto $J^{s+1}Y$. There is the exterior algebra homomorphism

$$h_0 : \phi_\lambda\, \mathrm{d}x^\lambda + \phi_i^\Lambda\, \mathrm{d}y^i_\Lambda \mapsto (\phi_\lambda + \phi_i^\Lambda\, y^i_{\lambda+\Lambda})\, \mathrm{d}x^\lambda \tag{8}$$

called the horizontal projection. It sends exterior forms on J^sY onto the horizontal forms on $J^{s+1}Y \to X$, and vanishes on the contact forms $\theta^i_\Lambda = \mathrm{d}y^i_\Lambda - y^i_{\lambda+\Lambda}\, \mathrm{d}x^\lambda$. Recall also the total derivative $d_\lambda = \partial_\lambda + y^i_{\lambda+\Lambda}\partial_i^\Lambda$ and the horizontal differential $\mathrm{d}_H\phi = \mathrm{d}x^\lambda \wedge d_\lambda\phi$ such that $h_0 \circ \mathrm{d} = \mathrm{d}_H \circ h_0$.

We regard a connection on a fibre bundle $Y \to X$ as a global section

$$\Gamma = \mathrm{d}x^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) \tag{9}$$

of the affine jet bundle $\pi^1_0 : J^1Y \to Y$. Sections of the underlying vector bundle $T^*X \underset{Y}{\otimes} VY \to Y$ are called soldering forms.

3. Lagrangian dynamics

Given a Lagrangian L and its Lepagean equivalent H_L, the first variational formula of the calculus of variations provides the canonical decomposition of the Lie derivative of L along a projectable vector field u on Y:

$$\mathbf{L}_{J^1u} L = u_V \rfloor \mathcal{E}_L + \mathrm{d}_H h_0\,(u \rfloor H_L) \tag{10}$$

where $u_V = (u\rfloor\theta^i)\partial_i$ and

$$\mathcal{E}_L = (\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L}\theta^i \wedge \omega : J^2Y \to T^*Y \wedge (\overset{n}{\wedge} T^*X) \tag{11}$$

is the Euler–Lagrange operator. Its kernel is the Euler–Lagrange equation on Y

$$(\partial_i - d_\lambda \partial_i^\lambda)\mathcal{L} = 0. \tag{12}$$

We will restrict our consideration to the Poincaré–Cartan form

$$H_L = L + \pi_i^\lambda \theta^i \wedge \omega_\lambda \qquad \pi_i^\lambda = \partial_i^\lambda \mathcal{L}. \tag{13}$$

Being a Lepagean equivalent of the Lagrangian $L = h_0(H_L)$ on J^1Y, this is also a Lepagean equivalent of the Lagrangian

$$\overline{L} = \hat{h}_0(H_L) = (\mathcal{L} + (\hat{y}^i_\lambda - y^i_\lambda)\pi^\lambda_i)\omega \qquad \hat{h}_0(\mathrm{d}y^i) = \hat{y}^i_\lambda\,\mathrm{d}x^\lambda \tag{14}$$

on the repeated jet manifold J^1J^1Y. The Euler–Lagrange operator for $\overline{L}$ reads

$$\begin{aligned} &\mathcal{E}_{\overline{L}} : J^1J^1Y \to T^*J^1Y \wedge (\overset{n}{\wedge} T^*X) \\ &\mathcal{E}_{\overline{L}} = [(\partial_i\mathcal{L} - \hat{d}_\lambda\pi^\lambda_i + \partial_i\pi^\lambda_j(\hat{y}^j_\lambda - y^j_\lambda))\,\mathrm{d}y^i + \partial^\lambda_i\pi^\mu_j(\hat{y}^j_\mu - y^j_\mu)\,\mathrm{d}y^i_\lambda] \wedge \omega \\ &\hat{d}_\lambda = \partial_\lambda + \hat{y}^i_\lambda\partial_i + y^i_{\lambda\mu}\partial^\mu_i. \end{aligned} \tag{15}$$

Its kernel Ker $\mathcal{E}_{\overline{L}} \subset J^1J^1Y$ is the Cartan equations

$$\partial^\lambda_i\pi^\mu_j(\hat{y}^j_\mu - y^j_\mu) = 0 \tag{16a}$$

$$\partial_i\mathcal{L} - \hat{d}_\lambda\pi^\lambda_i + (\hat{y}^j_\lambda - y^j_\lambda)\partial_i\pi^\lambda_j = 0. \tag{16b}$$

Since $\mathcal{E}_{\overline{L}}|_{J^2Y} = \mathcal{E}_L$, the Cartan equations (16*a*), (16*b*) are equivalent to the Euler–Lagrange equations (12) on integrable sections of $J^1Y \to X$. These equations are equivalent in the case of regular Lagrangians. On sections $\overline{s} : X \to J^1Y$, the Cartan equations (16*a*), (16*b*) are equivalent to the relation

$$\overline{s}^*(u\rfloor\mathrm{d}H_L) = 0 \tag{17}$$

which is assumed to hold for all vertical vector fields u on $J^1Y \to X$.

The Poincaré–Cartan form H_L (13) yields the above-mentioned Legendre morphism

$$\hat{H}_L : J^1Y \underset{Y}{\to} Z_Y \qquad (p^\mu_i, p) \circ \hat{H}_L = (\pi^\mu_i, \mathcal{L} - \pi^\mu_i y^i_\mu)$$

where the bundle Z_Y (2) is equipped with holonomic coordinates $(x^\lambda, y^i, p^\lambda_i, p)$. Due to the monomorphism $Z_Y \hookrightarrow \overset{n}{\wedge} T^*Y$, the bundle Z_Y is endowed with the pull-back

$$\Xi_Y = p\omega + p^\lambda_i\,\mathrm{d}y^i \wedge \omega_\lambda \tag{18}$$

of the canonical form Θ on $\overset{n}{\wedge} T^*Y$ whose exterior differential $\mathrm{d}\Theta$ is the n-multisymplectic form in the sense of Martin [1, 8].

Let $Z_L = \hat{H}_L(J^1Y)$ be an embedded subbundle $i_L : Z_L \hookrightarrow Z_Y$ of $Z_Y \to Y$. It is provided with the pull-back De Donder form $\Xi_L = i^*_L\Xi_Y$. We have

$$H_L = \hat{H}^*_L\Xi_L = \hat{H}^*_L(i^*_L\Xi_Y). \tag{19}$$

By analogy with the Cartan equations (17), the Hamilton–De Donder equations for sections $\overline{r}$ of $Z_L \to X$ are written as

$$\overline{r}^*(u\rfloor\mathrm{d}\Xi_L) = 0 \tag{20}$$

where u is an arbitrary vertical vector field on $Z_L \to X$.

Theorem 1 ([5]). *Let the Legendre morphism $\hat{H}_L$ be a submersion. Then a section $\overline{s}$ of $J^1Y \to X$ is a solution of the Cartan equations (17) iff $\hat{H}_L \circ \overline{s}$ is a solution of the Hamilton–De Donder equations (20); i.e., as was mentioned above, the Cartan and Hamilton–De Donder equations are quasi-equivalent.*

4. Covariant Hamiltonian dynamics

Given a fibre bundle $Y \to X$, let

$$\pi_{\Pi X} = \pi \circ \pi_{\Pi Y} : \Pi \to Y \to X$$

be the Legendre bundle (3). The canonical polysymplectic form, Hamiltonian connections and Hamiltonian forms are the main ingredients in the covariant Hamiltonian dynamics on Π (see [2, 4, 12] for a detailed exposition).

Let us consider the canonical bundle monomorphism

$$\theta = -p_i^\lambda \, \mathrm{d}y^i \wedge \omega \otimes \partial_\lambda : \Pi \underset{Y}{\hookrightarrow} \overset{n+1}{\wedge} T^*Y \underset{Y}{\otimes} TX. \tag{21}$$

The polysymplectic form on Π is defined as a unique TX-valued $(n+2)$-form

$$\Omega = \mathrm{d}p_i^\lambda \wedge \mathrm{d}y^i \wedge \omega \otimes \partial_\lambda \tag{22}$$

such that the relation $\Omega \rfloor \phi = -\mathrm{d}(\theta \rfloor \phi)$ holds for any exterior one-form ϕ on X. A connection

$$\gamma = \mathrm{d}x^\lambda \otimes (\partial_\lambda + \gamma_\lambda^i \partial_i + \gamma_{\lambda i}^\mu \partial_\mu^i) \tag{23}$$

on $\Pi \to X$ is called a Hamiltonian connection if the exterior form $\gamma \rfloor \Omega$ is closed. A Hamiltonian form H on Π has been defined above as the pull-back $h^* \Xi_Y$ (5) of the canonical form Ξ_Y (18) by a section h of the fibre bundle (4).

Theorem 2 ([10, 12]). *For every Hamiltonian form H (5), there exists an associated Hamiltonian connection such that*

$$\gamma \rfloor \Omega = \mathrm{d}H \qquad \gamma_\lambda^i = \partial_\lambda^i \mathcal{H} \qquad \gamma_{\lambda i}^\lambda = -\partial_i \mathcal{H}. \tag{24}$$

Conversely, for any Hamiltonian connection γ, there exists a local Hamiltonian form H on a neighbourhood of any point $q \in \Pi$ such that the relation (24) holds.

Hamiltonian forms on Π constitute an affine space modelled over the linear space of horizontal densities $\tilde{H} = \tilde{\mathcal{H}}\omega$ on $\Pi \to X$. This is an immediate consequence of the fact that (4) is an affine bundle modelled over the pull-back vector bundle $\Pi \underset{X}{\times} \overset{n}{\wedge} T^*X \to \Pi$. Every connection Γ (9) on $Y \to X$ defines the section

$$h_\Gamma : \overline{\mathrm{d}}y^i \mapsto \mathrm{d}y^i - \Gamma_\lambda^i \mathrm{d}x^\lambda$$

of the affine bundle $Z_Y \to \Pi$ and the corresponding Hamiltonian form

$$H_\Gamma = h_\Gamma^* \Xi_Y = p_i^\lambda \mathrm{d}y^i \wedge \omega_\lambda - p_i^\lambda \Gamma_\lambda^i \omega. \tag{25}$$

As a consequence, given a connection Γ on $Y \to X$, every Hamiltonian form H admits the decomposition

$$H = H_\Gamma - \tilde{H}_\Gamma = p_i^\lambda \mathrm{d}y^i \wedge \omega_\lambda - p_i^\lambda \Gamma_\lambda^i \omega - \tilde{\mathcal{H}}_\Gamma \omega. \tag{26}$$

Any bundle morphism

$$\Phi = \mathrm{d}x^\lambda \otimes (\partial_\lambda + \Phi_\lambda^i \partial_i) : \Pi \underset{Y}{\to} J^1Y \tag{27}$$

called a Hamiltonian map, defines the Hamiltonian form

$$H_\Phi = \Phi \rfloor \theta = p_i^\lambda \mathrm{d}y^i \wedge \omega_\lambda - p_i^\lambda \Phi_\lambda^i \omega. \tag{28}$$

Every Hamiltonian form H (5) yields the Hamiltonian map $\hat{H}$ such that

$$y_\lambda^i \circ \hat{H} = \partial_\lambda^i \mathcal{H}. \tag{29}$$

As was mentioned above, a Hamiltonian form H (5) on Π can be seen as the Poincaré–Cartan form of the Lagrangian $L_H = h_0(H)$ (6) on the jet manifold $J^1\Pi$. The Euler–Lagrange operator (11) for L_H, called the Hamilton operator for H, is

$$\begin{aligned}&\mathcal{E}_H : J^1\Pi \to T^*\Pi \wedge (\overset{n}{\wedge} T^*X)\\&\mathcal{E}_H = [(y^i_\lambda - \partial^i_\lambda\mathcal{H})\,\mathrm{d}p^\lambda_i - (p^\lambda_{\lambda i} + \partial_i\mathcal{H})\,\mathrm{d}y^i] \wedge \omega.\end{aligned} \tag{30}$$

Its kernel is the covariant Hamilton equations

$$y^i_\lambda = \partial^i_\lambda\mathcal{H} \tag{31a}$$

$$p^\lambda_{\lambda i} = -\partial_i\mathcal{H}. \tag{31b}$$

It is readily observed that all Hamiltonian connections (24) associated with a Hamiltonian form H live in the kernel of the Hamilton operator $\mathcal{E}_H$. Consequently, every integral section $J^1r = \gamma\circ r$ of a Hamiltonian connection γ associated with a Hamiltonian form H is a solution of the Hamilton equations (31*a*), (31*b*).

Note that, similarly to the Cartan equations (17), the Hamilton equations (31*a*), (31*b*) are equivalent to the condition

$$r^*(u\rfloor \mathrm{d}H) = 0 \tag{32}$$

for any vertical vector field u on $\Pi \to X$. The Hamilton equation (31*a*) can also be written as the equality

$$J^1(\pi_{\Pi Y}\circ r) = \hat{H}\circ r. \tag{33}$$

5. Lagrangian and Hamiltonian degenerate systems

Let us study the relations between Hamilton and Euler–Lagrange equations when a Lagrangian is degenerate. Their main peculiarity is that there is a set of Hamiltonian forms associated with the same degenerate Lagrangian.

Given a Lagrangian L, let

$$\hat{L} : J^1Y \underset{Y}{\to} \Pi \qquad p^\lambda_i\circ\hat{L} = \pi^\lambda_i$$

be the corresponding Legendre map. A Hamiltonian form H is said to be associated with a Lagrangian L if H satisfies the relations

$$\hat{L}\circ\hat{H}\circ\hat{L} = \hat{L} \tag{34a}$$

$$H = H_{\hat{H}} + \hat{H}^*L. \tag{34b}$$

A glance at the relation (34*a*) shows that $\hat{L}\circ\hat{H}$ is the projector

$$p^\mu_i(q) = \partial^\mu_i\mathcal{L}(x^\mu, y^i, \partial^j_\lambda\mathcal{H}(q)) \qquad q\in N_L \tag{35}$$

from Π onto the Lagrangian constraint space $N_L = \hat{L}(J^1Y)$. Accordingly, $\hat{H}\circ\hat{L}$ is the projector from J^1Y onto $\hat{H}(N_L)$. A Hamiltonian form is called weakly associated with a Lagrangian L if the condition (34*b*) holds on the Lagrangian constraint space N_L. The following assertions take place [4, 10].

Proposition 3. *If a Hamiltonian map Φ (27) obeys the relation (34a), then the Hamiltonian form $H = H_\phi + \Phi^*L$ is weakly associated with the Lagrangian L. If $\Phi = \hat{H}$, then H is associated with L.*

Proposition 4. *Any Hamiltonian form H weakly associated with a Lagrangian L obeys the relation*

$$H|_{N_L} = \hat{H}^*H_L|_{N_L} \tag{36}$$

where H_L is the Poincaré–Cartan form (13).

The difference between associated and weakly associated Hamiltonian forms lies in the following. Let H be an associated Hamiltonian form, i.e., the equality (34*b*) holds everywhere on Π. It takes the coordinate form

$$\mathcal{H}(q) = p_i^\mu \partial_\mu^i \mathcal{H} - \mathcal{L}(x^\mu, y^j, \partial_\lambda^j \mathcal{H}(q)) \qquad q \in N_L.$$

Acting on this equality by the exterior differential, we obtain the relations

$$\partial_\mu \mathcal{H}(q) = -(\partial_\mu \mathcal{L}) \circ \hat{H}(q) \qquad \partial_i \mathcal{H}(q) = -(\partial_i \mathcal{L}) \circ \hat{H}(q) \qquad q \in N_L$$
$$(p_i^\mu - (\partial_i^\mu \mathcal{L})(x^\mu, y^j, \partial_\lambda^j \mathcal{H}))\partial_\mu^i \partial_\alpha^a \mathcal{H} = 0.$$

The last of these shows that the associated Hamiltonian form is not regular outside the Lagrangian constraint space N_L. In particular, any Hamiltonian form is weakly associated with the Lagrangian $L = 0$, while the associated Hamiltonian forms are only of the form H_Γ (25).

Something more can be said in the case of semiregular Lagrangians. A Lagrangian L is called semiregular if the pre-image $\hat{L}^{-1}(q)$ of any point $q \in N_L$ is a connected submanifold of J^1Y. We now introduce the following lemma [10, 15].

Lemma 5. *The Poincaré–Cartan form H_L for a semiregular Lagrangian L is constant on the connected pre-image $\hat{L}^{-1}(q)$ of any point $q \in N_L$.*

Corollary 6. *All Hamiltonian forms weakly associated with a semiregular Lagrangian L coincide with each other on the Lagrangian constraint space N_L, and the Poincaré–Cartan form H_L (13) for L is the pull-back*

$$H_L = \hat{L}^* H \qquad (\pi_i^\lambda y_\lambda^i - \mathcal{L})\omega = \mathcal{H}(x^\mu, y^j, \pi_j^\mu)\omega \tag{37}$$

of any such a Hamiltonian form H.

Corollary 6 enables us to connect Euler–Lagrange and Cartan equations for a semiregular Lagrangian L with Hamilton equations for Hamiltonian forms weakly associated with L.

Theorem 7. *Let a section r of $\Pi \to X$ be a solution of the Hamilton equations (31a), (31b) for a Hamiltonian form H weakly associated with a semiregular Lagrangian L. If r lives in the constraint space N_L, the section $s = \pi_{\Pi Y} \circ r$ of $Y \to X$ satisfies the Euler–Lagrange equations (12), while $\overline{s} = \hat{H} \circ r$ obeys the Cartan equations (16a), (16b).*

Proof. Acting by the exterior differential on the relation (37), we obtain the relation

$$(y_\lambda^i - \partial_\lambda^i \mathcal{H} \circ \hat{L})\, \mathrm{d}\pi_i^\lambda \wedge \omega - (\partial_i \mathcal{L} + \partial_i(\mathcal{H} \circ \hat{L}))\, \mathrm{d}y^i \wedge \omega = 0 \tag{38}$$

which is equivalent to the system of equalities

$$\partial_i^\lambda \pi_j^\mu (y_\mu^j - \partial_\mu^j \mathcal{H} \circ \hat{L}) = 0 \qquad \partial_i \pi_j^\mu (y_\mu^j - \partial_\mu^j \mathcal{H} \circ \hat{L}) - (\partial_i \mathcal{L} + (\partial_i \mathcal{H}) \circ \hat{L}) = 0.$$

Using these equalities and the relation $(p_i^\lambda, y_\mu^i, p_{\mu i}^\lambda) \circ J^1\hat{L} = (\pi_i^\lambda, \hat{y}_\mu^i, \hat{d}_\mu \pi_i^\lambda)$, one can easily see that $\mathcal{E}_{\overline{L}} = (J^1\hat{L})^* \mathcal{E}_H$, where $\mathcal{E}_{\overline{L}}$ is the Euler–Lagrange–Cartan operator (15). Let r be a section of $\Pi \to X$ which lives in the Lagrangian constraint space N_L, and $\overline{s} = \hat{H} \circ r$. Then we have

$$r = \hat{L} \circ \overline{s} \qquad J^1 r = J^1\hat{L} \circ J^1\overline{s}.$$

If r is a solution of the Hamilton equations, the exterior form $\mathcal{E}_H$ vanishes on $J^1r(X)$. Hence, the pull-back form $\mathcal{E}_{\overline{L}} = (J^1\hat{L})^* \mathcal{E}_H$ vanishes on $J^1\overline{s}(X)$. It follows that $\overline{s}$ obeys the Cartan equations (16*a*), (16*b*). We obtain from the equality (33) that $\overline{s} = J^1 s$, $s = \pi_{\Pi Y} \circ r$. Hence, s is a solution of the Euler–Lagrange equations. □

The same result can be obtained from the relation

$$\overline{L} = (J^1\hat{L})^* L_H \tag{39}$$

where $\overline{L}$ is the Lagrangian (14) on J^1J^1Y and L_H is the Lagrangian (6) on $J^1\Pi$.

Theorem 8. *Given a semiregular Lagrangian L, let a section $\overline{s}$ of the jet bundle $J^1Y \to X$ be a solution of the Cartan equations (16a), (16b). Let H be a Hamiltonian form weakly associated with L, and let H satisfy the relation*

$$\hat{H} \circ \hat{L} \circ \overline{s} = J^1(\pi^1_0 \circ \overline{s}). \tag{40}$$

Then, the section $r = \hat{L} \circ \overline{s}$ of the Legendre bundle $\Pi \to X$ is a solution of the Hamilton equations (31a), (31b) for H.

Proof. The Hamilton equations (31*a*) hold by virtue of the condition (40). Substituting $\hat{L} \circ \overline{s}$ in the Hamilton equations (31*b*) and using the relations (38) and (40), we come to the Cartan equations (16*b*) for $\overline{s}$ as follows:

$$\begin{aligned}\hat{d}_\lambda \pi^\lambda_i \circ \overline{s} + (\partial_i \mathcal{H}) \circ \hat{L} \circ \overline{s} &= \hat{d}_\lambda \pi^\lambda_i \circ \overline{s} + (\overline{s}^j_\mu - \partial^j_\mu \mathcal{H} \circ \hat{L} \circ \overline{s}) \partial_i \pi^\mu_j \circ \overline{s} - \partial_i \mathcal{L} \circ \overline{s} \\ &= \hat{d}_\lambda \pi^\lambda_i \circ \overline{s} - (\partial_\mu \overline{s}^j - \overline{s}^j_\mu) \partial_i \pi^\mu_j \circ \overline{s} - \partial_i \mathcal{L} \circ \overline{s} = 0.\end{aligned}$$

□

Remark 1. *Since $\hat{H} \circ \hat{L}$ in theorem 8 is a projection operator, the condition (40) implies that the solution $\overline{s}$ of the Cartan equations is actually an integrable section $\overline{s} = J^1 s$ where s is a solution of the Euler–Lagrange equations. Theorems 7 and 8 show that, if a solution of the Cartan equations provides a solution of the covariant Hamilton equations, it is necessarily a solution of the Euler–Lagrange equations.*

We will say that a set of Hamiltonian forms H weakly associated with a semiregular Lagrangian L is complete if, for each solution s of the Euler–Lagrange equations, there exists a solution r of the Hamilton equations for a Hamiltonian form H from this set such that $s = \pi_{\Pi Y} \circ r$. By virtue of theorem 8 and remark 1, a set of weakly associated Hamiltonian forms is complete if, for every solution s on X of the Euler–Lagrange equations for L, there is a Hamiltonian form H from this set which fulfils the relation (7).

As for the existence of complete sets of weakly associated Hamiltonian forms, we refer to the following theorem. A Lagrangian L is said to be almost regular if: (i) L is semiregular, (ii) the Lagrangian constraint space N_L is a closed imbedded subbundle $i_N : N_L \hookrightarrow \Pi$ of the Legendre bundle $\Pi \to Y$; (iii) the Legendre map

$$\hat{L} : J^1Y \to N_L \tag{41}$$

is a submersion, i.e., a fibred manifold.

Proposition 9 ([12, 15]). *Let L be an almost-regular Lagrangian. On an open neighbourhood in Π of each point $q \in N_L$, there exist local Hamiltonian forms associated with L which constitute a complete set.*

In the case of an almost-regular Lagrangian L, we can say something more on the relations between Lagrangian and Hamiltonian systems as follows. Let us assume that the fibred manifold (41) admits a global section Ψ. Let us consider the pull-back

$$H_N = \Psi^* H_L \tag{42}$$

called the constrained Hamiltonian form [3, 4]. By virtue of lemma 5, it does not depend on the choice of a section of the fibred manifold $J^1Y \to N_L$, and so $H_L = \hat{L}^* H_N$. For sections r of the fibre bundle $N_L \to X$, we can write the constrained Hamilton equations as

$$r^*(u_N \rfloor \mathrm{d}H_N) = 0 \tag{43}$$

where u_N is an arbitrary vertical vector field on $N_L \to X$. These equations possess the following important properties.

Theorem 10. *For any Hamiltonian form H weakly associated with an almost-regular Lagrangian L, every solution r of the Hamilton equations which lives in the Lagrangian constraint space N_L is a solution of the constrained Hamilton equations (43).*

Proof. Such a Hamiltonian form H defines the global section $\Psi = \hat{H} \circ i_N$ of the fibred manifold (41). Due to the relation (37), $H_N = i_N^* H$ and the constrained Hamilton equations can be written as

$$r^*(u_N \rfloor \mathrm{d}i_N^* H) = r^*(u_N \rfloor \mathrm{d}H|_{N_L}) = 0. \tag{44}$$

They differ from the Hamilton equations (32) restricted to N_L which read

$$r^*(u \rfloor \mathrm{d}H|_{N_L}) = 0 \tag{45}$$

where r is a section of $N_L \to X$ and u is an arbitrary vertical vector field on $\Pi \to X$. A solution r of the equations (45) obviously satisfies the weaker condition (44). □

Theorem 11. *The constrained Hamilton equations (43) are equivalent to the Hamilton–De Donder equations (20).*

Proof. It is readily seen that $\hat{L} = \pi_{Z\Pi} \circ \hat{H}_L$. Hence, the projection $\pi_{Z\Pi}$ (4) yields a surjection of Z_L onto N_L. Given a section Ψ of the fibred manifold (41), we have the morphism $\hat{H}_L \circ \Psi : N_L \to Z_L$. By virtue of lemma 5, this is a surjection such that

$$\pi_{Z\Pi} \circ \hat{H}_L \circ \Psi = \mathrm{Id}\,_{N_L}.$$

Hence, $\hat{H}_L \circ \Psi$ is a bundle isomorphism over Y which is independent of the choice of a global section Ψ. Combining (19) and (42) gives $H_N = (\hat{H}_L \circ \Psi)^* \Xi_L$ that leads to the desired equivalence. □

The above proof gives more. Namely, since Z_L and N_L are isomorphic, the Legendre morphism H_L fulfils the conditions of theorem 1. Then combining theorems 1 and 11, we obtain the following theorem.

Theorem 12. *Let L be an almost-regular Lagrangian such that the fibred manifold (41) has a global section. A section $\overline{s}$ of the jet bundle $J^1Y \to X$ is a solution of the Cartan equations (17) iff $\hat{L} \circ \overline{s}$ is a solution of the constrained Hamilton equations (43).*

Theorem 12 is also a corollary of lemma 13 below. The constrained Hamiltonian form H_N (42) defines the constrained Lagrangian

$$L_N = h_0(H_N) = (J^1 i_N)^* L_H \tag{46}$$

on the jet manifold $J^1 N_L$ of the fibre bundle $N_L \to X$.

Lemma 13. *There are the relations*

$$\overline{L} = (J^1\hat{L})^* L_N \qquad L_N = (J^1\Psi)^* \overline{L} \tag{47}$$

where $\overline{L}$ is the Lagrangian (14).

Proof. The first of the relations (47) is an immediate consequence of the relation (39). The latter follows from the relation $(y^i_\mu, \hat{y}^i_\lambda, y^i_{\lambda\mu}) \circ J^1\hat{H} = (\partial^i_\mu\mathcal{H}, y^i_\lambda, d_\lambda\partial^i_\mu\mathcal{H})$ and the relation (35) if we put $\Psi = \hat{H} \circ i_N$ for some Hamiltonian form H associated with the almost-regular Lagrangian L. □

The Euler–Lagrange equation for the constrained Lagrangian L_N (46) are equivalent to the constrained Hamilton equations and, by virtue of lemma 13, are quasi-equivalent to the Cartan equations. At the same time, Cartan equations of degenerate Lagrangian systems contain an additional freedom in comparison with the restricted Hamilton equations (see the next section).

6. Quadratic degenerate systems

This section is devoted to the physically important case of almost-regular quadratic Lagrangians. Given a fibre bundle $Y \to X$, let us consider a quadratic Lagrangian L which has the coordinate expression

$$\mathcal{L} = \tfrac{1}{2}a^{\lambda\mu}_{ij}y^i_\lambda y^j_\mu + b^\lambda_i y^i_\lambda + c \tag{48}$$

where a, b and c are local functions on Y. This property is coordinate-independent due to the affine transformation law of coordinates y^i_λ. The associated Legendre map

$$p^\lambda_i \circ \hat{L} = a^{\lambda\mu}_{ij}y^j_\mu + b^\lambda_i \tag{49}$$

is an affine morphism over Y. It defines the corresponding linear morphism

$$\overline{L} : T^*X \underset{Y}{\otimes} VY \underset{Y}{\to} \Pi \qquad p^\lambda_i \circ \overline{L} = a^{\lambda\mu}_{ij}\overline{y}^j_\mu \tag{50}$$

where $\overline{y}^j_\mu$ are bundle coordinates on the vector bundle $T^*X \underset{Y}{\otimes} VY$.

Let the Lagrangian L (48) be almost regular, i.e. the matrix function $a^{\lambda\mu}_{ij}$ is of constant rank. Then the Lagrangian constraint space N_L (49) is an affine subbundle of the Legendre bundle $\Pi \to Y$, modelled over the vector subbundle $\overline{N}_L$ (50) of $\Pi \to Y$. Hence, $N_L \to Y$ has a global section. For the sake of simplicity, let us assume that it is the canonical zero section $\hat{0}(Y)$ of $\Pi \to Y$. Then $\overline{N}_L = N_L$. Accordingly, the kernel of the Legendre map (49) is an affine subbundle of the affine jet bundle $J^1Y \to Y$, modelled over the kernel of the linear morphism $\overline{L}$ (50). Then there exists a connection

$$\Gamma : Y \to \mathrm{Ker}\,\hat{L} \subset J^1Y \tag{51}$$

$$a^{\lambda\mu}_{ij}\Gamma^j_\mu + b^\lambda_i = 0 \tag{52}$$

on $Y \to X$. Connections (51) constitute an affine space modelled over the linear space of soldering forms ϕ on $Y \to X$ satisfying the conditions

$$a^{\lambda\mu}_{ij}\phi^j_\mu = 0 \tag{53}$$

and, as a consequence, the conditions $\phi^i_\lambda b^\lambda_i = 0$. If the Lagrangian (48) is regular, the connection (51) is unique.

The matrix a in the Lagrangian L (48) can be seen as a global section of constant rank of the tensor bundle

$$\overset{n}{\wedge} T^*X \underset{Y}{\otimes} [\overset{2}{\vee}(TX \underset{Y}{\otimes} V^*Y)] \to Y.$$

Then it satisfies the following corollary of the well known theorem on the splitting of an exact sequence of vector bundles.

Lemma 14. *Given a k-dimensional vector bundle* $E \to Z$*, let* a *be a section of rank* r *of the tensor bundle* $\overset{2}{\vee} E^* \to Z$*. There is a splitting* $E = \mathrm{Ker}\ a \underset{Z}{\oplus} E'$ *where* $E' = E/\mathrm{Ker}\ a$ *is the quotient bundle, and* a *is a non-degenerate section of* $\overset{2}{\vee} E'^* \to Z$*.*

Theorem 15. *There exists a linear bundle map*

$$\sigma : \Pi \underset{Y}{\to} T^*X \underset{Y}{\otimes} VY \qquad \overline{y}^i_\lambda \circ \sigma = \sigma^{ij}_{\lambda\mu} p^\mu_j \tag{54}$$

such that $\overline{L} \circ \sigma \circ i_N = i_N$*.*

Proof. The map (54) is a solution of the algebraic equations

$$a^{\lambda\mu}_{ij} \sigma^{jk}_{\mu\alpha} a^{\alpha\nu}_{kb} = a^{\lambda\nu}_{ib}. \tag{55}$$

By virtue of lemma 14, there exists the bundle slitting

$$T^*X \underset{Y}{\otimes} VY = \mathrm{Ker}\ a \underset{Y}{\oplus} E' \tag{56}$$

and a (non-holonomic) atlas of this bundle such that transition functions of Ker a and E' are independent. Since a is a non-degenerate section of $\overset{n}{\wedge} T^*X \underset{Y}{\otimes} (\overset{2}{\vee} E'^*) \to Y$, there exists an atlas of E' such that a is brought into a diagonal matrix with non-vanishing components a^{AA}. Due to the splitting (56), we have the corresponding bundle splitting

$$TX \underset{Y}{\otimes} V^*Y = (\mathrm{Ker}\ a)^* \underset{Y}{\oplus} E'^*.$$

Then the desired map σ is represented by a direct sum $\sigma_1 \oplus \sigma_0$ of an arbitrary section σ_1 of the fibre bundle

$$\overset{n}{\wedge} TX \underset{Y}{\otimes} (\overset{2}{\vee} \mathrm{Ker}\ a) \to Y$$

and the section σ_0 of the fibre bundle

$$\overset{n}{\wedge} TX \underset{Y}{\otimes} (\overset{2}{\vee} E') \to Y$$

which has non-vanishing components $\sigma_{AA} = (a^{AA})^{-1}$ with respect to the above-mentioned atlas of E'. Moreover, σ satisfies the particular relations

$$\sigma_0 = \sigma_0 \circ \overline{L} \circ \sigma_0 \qquad a \circ \sigma_1 = 0 \qquad \sigma_1 \circ a = 0. \tag{57}$$

□

The following theorem is the key point of our consideration.

Theorem 16. *We have the splittings*

$$J^1Y = \mathcal{S}(J^1Y) \underset{Y}{\oplus} \mathcal{F}(J^1Y) = \mathrm{Ker}\ \hat{L} \underset{Y}{\oplus} \mathrm{Im}(\sigma \circ \hat{L}) \tag{58a}$$

$$y^i_\lambda = \mathcal{S}^i_\lambda + \mathcal{F}^i_\lambda = [y^i_\lambda - \sigma^{ik}_{\lambda\alpha}(a^{\alpha\mu}_{kj} y^j_\mu + b^\alpha_k)] + [\sigma^{ik}_{\lambda\alpha}(a^{\alpha\mu}_{kj} y^j_\mu + b^\alpha_k)] \tag{58b}$$

$$\Pi = \mathcal{R}(\Pi) \underset{Y}{\oplus} \mathcal{P}(\Pi) = \mathrm{Ker}\ \sigma_0 \underset{Y}{\oplus} N_L \tag{59a}$$

$$p^\lambda_i = \mathcal{R}^\lambda_i + \mathcal{P}^\lambda_i = [p^\lambda_i - a^{\lambda\mu}_{ij} \sigma^{jk}_{\mu\alpha} p^\alpha_k] + [a^{\lambda\mu}_{ij} \sigma^{jk}_{\mu\alpha} p^\alpha_k]. \tag{59b}$$

Proof. The proof follows from a direct computation by means of the relations (52), (55) and (57). □

It is readily observed that, with respect to the coordinates $\mathcal{S}^i_\lambda$ and $\mathcal{F}^i_\lambda$ (58*b*), the Lagrangian (48) reads

$$\mathcal{L} = \frac{1}{2} a^{\lambda\mu}_{ij} \mathcal{F}^i_\lambda \mathcal{F}^j_\mu + c' \tag{60}$$

while the Lagrangian constraint space is given by the reducible constraints

$$\mathcal{R}^\lambda_i = p^\lambda_i - a^{\lambda\mu}_{ij} \sigma^{jk}_{\mu\alpha} p^\alpha_k = 0. \tag{61}$$

Note that, in gauge theory, we have the canonical splitting (58*a*) where $2\mathcal{F}$ is the strength tensor [3,4,11]. The Yang–Mills Lagrangian of gauge theory is exactly of the form (60) where $c' = 0$. The Lagrangian of Proca fields is also of the form (60) where c' is the mass term. This is an example of a degenerate Lagrangian system without gauge symmetries.

Given the linear map σ (54) and a connection Γ (51), let us consider the affine Hamiltonian map

$$\Phi = \Gamma \circ \pi_{\Pi Y} + \sigma : \Pi \to J^1 Y \qquad \Phi^i_\lambda = \Gamma^i_\lambda + \sigma^{ij}_{\lambda\mu} p^\mu_j \tag{62}$$

and the Hamiltonian form

$$\begin{aligned} H = H_\Phi + \Phi^* L &= p^\lambda_i \, \mathrm{d}y^i \wedge \omega_\lambda - [\Gamma^i_\lambda p^\lambda_i + \tfrac{1}{2} \sigma_0{}^{ij}_{\lambda\mu} p^\lambda_i p^\mu_j + \sigma_1{}^{ij}_{\lambda\mu} p^\lambda_i p^\mu_j - c']\omega \\ &= (\mathcal{R}^\lambda_i + \mathcal{P}^\lambda_i) \, \mathrm{d}y^i \wedge \omega_\lambda - [(\mathcal{R}^\lambda_i + \mathcal{P}^\lambda_i)\Gamma^i_\lambda + \tfrac{1}{2} \sigma_0{}^{ij}_{\lambda\mu} \mathcal{P}^\lambda_i \mathcal{P}^\mu_j + \sigma_1{}^{ij}_{\lambda\mu} p^\lambda_i p^\mu_j - c']\omega. \end{aligned} \tag{63}$$

In particular, if σ_1 is non-degenerate, so is the Hamiltonian form (63).

Theorem 17. *The Hamiltonian forms (63) parametrized by connections Γ (51) are weakly associated with the Lagrangian (48) and constitute a complete set.*

Proof. By the very definitions of Γ and σ, the Hamiltonian map (62) satisfies the condition (34*a*). Then H is weakly associated with L (48) in accordance with proposition 3. Let us write the corresponding Hamilton equations (31*a*) for a section r of the Legendre bundle $\Pi \to X$. They are

$$J^1 s = (\Gamma \circ \pi_{\Pi Y} + \sigma) \circ r \qquad s = \pi_{\Pi Y} \circ r. \tag{64}$$

Due to the surjections $\mathcal{S}$ and $\mathcal{F}$ (58*a*), the Hamilton equations (64) break in two parts

$$\begin{aligned} &\mathcal{S} \circ J^1 s = \Gamma \circ s \\ &\partial_\lambda r^i - \sigma^{ik}_{\lambda\alpha} (a^{\alpha\mu}_{kj} \partial_\mu r^j + b^\alpha_k) = \Gamma^i_\lambda \circ s \end{aligned} \tag{65}$$

$$\begin{aligned} &\mathcal{F} \circ J^1 s = \sigma \circ r \\ &\sigma^{ik}_{\lambda\alpha} (a^{\alpha\mu}_{kj} \partial_\mu r^j + b^\alpha_k) = \sigma^{ik}_{\lambda\alpha} r^\alpha_k. \end{aligned} \tag{66}$$

Let s be an arbitrary section of $Y \to X$, e.g. a solution of the Euler–Lagrange equations. There exists a connection Γ (51) such that the relation (65) holds, namely $\Gamma = \mathcal{S} \circ \Gamma'$ where Γ' is a connection on $Y \to X$ which has s as an integral section. It is easily seen that, in this case, the Hamiltonian map (62) satisfies the relation (7) for s. Hence, the Hamiltonian forms (63) constitute a complete set. □

It is readily observed that, in the case $\sigma_1 = 0$, $\Phi = \hat{H}$ and the Hamiltonian forms (63) are associated with the Lagrangian (48). Thus, for different σ_1, we have different complete sets of Hamiltonian forms (63). Hamiltonian forms H (63) of such a complete set differ from each other in the term $\phi^i_\lambda \mathcal{R}^\lambda_i$, where ϕ are the soldering forms (53). It follows from the splitting (59*a*) that this term vanishes on the Lagrangian constraint space. The corresponding constrained Hamiltonian form $H_N = i^*_N H$ and the constrained Hamilton equations (43) can be written. In the case of quadratic Lagrangians, we can improve theorem 10 as follows.

Theorem 18. *For every Hamiltonian form H (63), the Hamilton equations (31b) and (66) restricted to the Lagrangian constraint space N_L are equivalent to the constrained Hamilton equations.*

Proof. Due to the splitting (59*a*), we have the corresponding splitting of the vertical tangent bundle $V_Y\Pi$ of the Legendre bundle $\Pi \to Y$. In particular, any vertical vector field u on $\Pi \to X$ admits the decomposition

$$u = [u - u_{TN}] + u_{TN} \qquad u_{TN} = u^i\partial_i + a^{\lambda\mu}_{ij}\sigma^{jk}_{\mu\alpha}u^\alpha_k\partial^i_\lambda$$

such that $u_N = u_{TN}|_{N_L}$ is a vertical vector field on the Lagrangian constraint space $N_L \to X$. Let us consider the equations

$$r^*(u_{TN}\rfloor \mathrm{d}H) = 0 \tag{67}$$

where r is a section of $\Pi \to X$ and u is an arbitrary vertical vector field on $\Pi \to X$. They are equivalent to the pair of equations

$$r^*(a^{\lambda\mu}_{ij}\sigma^{jk}_{\mu\alpha}\partial^i_\lambda\rfloor \mathrm{d}H) = 0 \tag{68a}$$

$$r^*(\partial_i\rfloor \mathrm{d}H) = 0. \tag{68b}$$

The equations (68*b*) are obviously the Hamilton equations (31*b*) for H. Bearing in mind the relations (52) and (57), one can easily show that equations (68*a*) coincide with the Hamilton equations (66). The proof is completed by observing that, restricted to the Lagrangian constraint space N_L, the equations (67) are exactly the constrained Hamilton equations (44). □

Note that, in Hamiltonian gauge theory, the restricted Hamiltonian form and the restricted Hamilton equations are gauge invariant.

Theorem 18 shows that, restricted to the Lagrangian constraint space, the Hamilton equations for different Hamiltonian forms (63) associated with the same quadratic Lagrangian (48) differ from each other in the equations (65). These equations are independent of momenta and play the role of gauge-type conditions as follows.

By virtue of theorem 12, the constrained Hamilton equation are quasi-equivalent to the Cartan equations. A section $\overline{s}$ of $J^1Y \to X$ is a solution of the Cartan equations for an almost-regular quadratic Lagrangian (48) iff $r = \hat{L}\circ\overline{s}$ is a solution of the Hamilton equations (31*b*) and (66). In particular, let $\overline{s}$ be such a solution of the Cartan equations and $\overline{s}_0$ a section of the fibre bundle $T^*X \underset{Y}{\otimes} VY \to X$ which takes its values into Ker $\overline{L}$ (see (50)) and projects onto the section $s = \pi^1_0\circ\overline{s}$ of $Y \to X$. Then the affine sum $\overline{s} + \overline{s}_0$ over $s(X) \subset Y$ is also a solution of the Cartan equations. Thus, we come to the notion of a gauge-type freedom of the Cartan equations for an almost-regular quadratic Lagrangian L. One can speak of the gauge classes of solutions of the Cartan equations whose elements differ from each other in the above-mentioned sections $\overline{s}_0$. Let z be such a gauge class whose elements project onto a section s of $Y \to X$. For different connections Γ (51), we consider the condition

$$\mathcal{S}\circ\overline{s} = \Gamma\circ s \qquad \overline{s}\in z. \tag{69}$$

Proposition 19. *(i) If two elements $\overline{s}$ and $\overline{s}'$ of the same gauge class z obey the same condition (69), then $\overline{s} = \overline{s}'$. (ii) For any solution $\overline{s}$ of the Cartan equations, there exists a connection (51) which fulfils condition (69).*

Proof. (i) Let us consider the affine difference $\overline{s} - \overline{s}'$ over $s(X) \subset Y$. We have $\mathcal{S}(\overline{s} - \overline{s}') = 0$ iff $\overline{s} = \overline{s}'$. (ii) In the proof of theorem 17, we have shown that, given $s = \pi^0_1\circ\overline{s}$, there exists a connection Γ (51) which fulfils the relation (65). Let us consider the affine difference

$\mathcal{S}(\overline{s} - J^1 s)$ over $s(X) \subset Y$. This is a local section of the vector bundle Ker $\overline{L} \to Y$ over $s(X)$. Let ϕ be its prolongation onto Y. It is easy to see that $\Gamma + \phi$ is the desired connection. □

Due to the properties in proposition 19, one can treat (69) as a gauge-type condition on solutions of the Cartan equations. The Hamilton equations (65) exemplify this gauge-type condition when $\overline{s} = J^1 s$ is a solution of the Euler–Lagrange equations. At the same time, the above-mentioned freedom characterizes solutions of the Cartan equations, but not of the Euler–Lagrange ones. First of all, this freedom reflects the degeneracy of the Cartan equations (16*a*). Therefore, in the Hamiltonian gauge theory, the above freedom is not related directly to the familiar gauge invariance. Nevertheless, the Hamilton equations (65) are not gauge invariant, and also can play the role of gauge conditions in gauge theory.

References

[1] Cantrijn F, Ibort L and De León M 1996 *Rend. Sem. Mat. Univ. Pol. Torino* **54** 225
[2] Cariñena J, Crampin M and Ibort L 1991 *Differ. Geom. Appl.* **1** 345
[3] Giachetta G and Mangiarotti L 1995 *Int. J. Theor. Phys.* **34** 2353
[4] Giachetta G, Mangiarotti L and Sardanashvily G 1997 *New Lagrangian and Hamiltonian Methods in Field Theory* (Singapore: World Scientific)
[5] Gotay M 1991 *Mechanics, Analysis and Geometry: 200 Years after Lagrange* ed M Francaviglia (Amsterdam: North-Holland) pp 203–35
[6] Günther C 1987 *J. Differ. Geom.* **25** 23
[7] Kijowski J and Tulczyjew W 1979 *A Symplectic Framework for Field Theories* (Berlin: Springer)
[8] Martin G 1988 *Lett. Math. Phys.* **16** 133
[9] Sardanashvily G and Zakharov O 1993 *Diff. Geom. Appl.* **3** 245
[10] Sardanashvily G 1993 *Gauge Theory in Jet Manifolds* (Palm Harbour, FL: Hadronic)
[11] Sardanashvily G 1994 *J. Math. Phys.* **35** 6584
[12] Sardanashvily G 1995 *Generalized Hamiltonian Formalism for Field Theory. Constraint Systems* (Singapore: World Scientific)
[13] Sardanashvily G 1997 *J. Math. Phys.* **38** 847
[14] Saunders D 1989 *The Geometry of Jet Bundles* (Cambridge: Cambridge University Press)
[15] Zakharov O 1992 *J. Math. Phys.* **33** 607